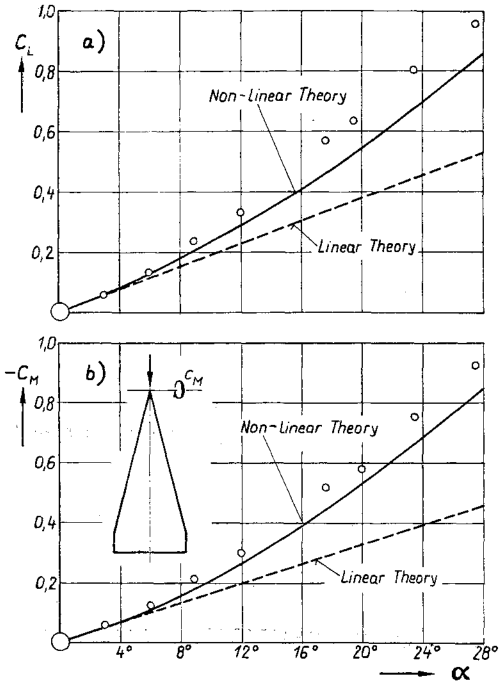

Fig. 2 Lift coefficient a) and pitching-moment coefficient b) for a slender delta wing (aspect ratio AR = 0.78, taper ratio λ = 0.125) vs angle of attack

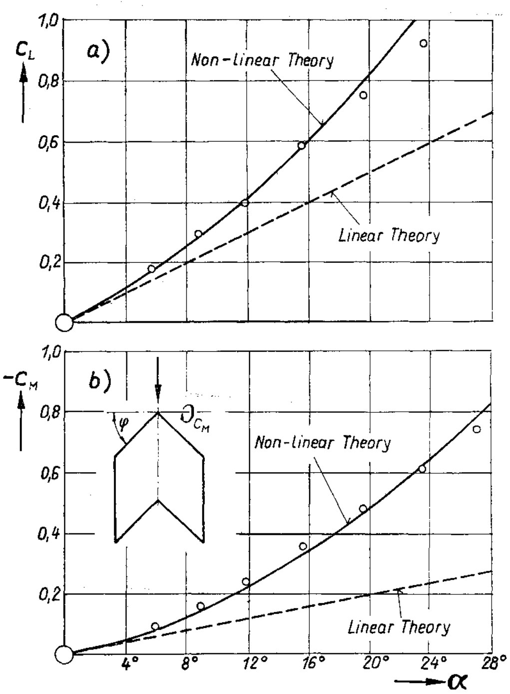

Fig. 3 Lift coefficient a) and pitching-moment coefficient b) for a swept wing (aspect ratio AR = 1, sweep angle φ = 45°) vs angle of attack

wings, and delta wings were calculated and, as far as possible, compared with measurements. The agreement is quite satisfactory. In Figs. 2 and 3, comparisons between theory and experiments are shown for a slender delta wing and for a swept wing of 45° sweep angle and taper ratio 1. For the lift coefficients as well as for the pitching-moment coefficients, the agreement is very good up to high values of angle of attack. For the drag coefficient of wings having sharp leading edges, one obtains $c_D = c_L\alpha$, since suction forces are zero due to leading-edge separation.

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Cylindrical Heat Flow with Arbitrary Heating Rates

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In this note, Chen's solution[1] is extended to the problem of purely radial heat flow through a hollow cylinder ($a \leqslant r \leqslant b$) under an arbitrary time-dependent heat flux at the outer surface ($r = b$) and zero heat flux at the internal boundary ($r = a$). The solution should be useful in current aerospace problems for stations of a missile body not influenced by nose tapering. The missile's skin material is assumed to have physical properties independent of temperature, so that the temperature $T(r,t)$ is a function of radius r and time t only.

The basic differential equation and boundary conditions can be written in the form

$$\frac{1}{\alpha}\frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} \tag{1}$$

with

$$T(r,0) = 0 \tag{2}$$

for $t = 0$, $a \leqslant r \leqslant b$, and

$$k\,\frac{\partial T(b,t)}{\partial r} = Q(t) \qquad k\,\frac{\partial T(a,t)}{\partial t} = 0 \tag{3}$$

where $Q(t)$ is the heat flux at the external boundary. Using the Laplace transform,

$$\bar{T}(r,p) = \int_0^\infty e^{-pt}T(r,t)dt = \mathcal{L}\{T(r,t)\} \tag{4}$$

these equations become

$$q^2\bar{T} = \frac{\partial^2\bar{T}}{\partial r^2} + \frac{1}{r}\frac{\partial \bar{T}}{\partial r} \qquad (\alpha q^2 = p) \tag{5}$$

with

$$k[\partial\bar{T}(b,p)/\partial r] = \bar{Q}(p) \tag{6}$$

and

$$k[\partial\bar{T}(a,p)/\partial r] = 0 \tag{7}$$

The operator form of the solution is

$$\bar{T}(r,p) = \frac{\bar{Q}(p)[I_0(qr)K_1(qa) + K_0(qr)I_1(qa)]}{kq[I_1(qb)K_1(qa) - I_1(qa)K_1(qb)]} \tag{8}$$

where I_0, I_1, K_0, K_1 are modified Bessel functions of the first

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and second kinds.

For $Q(t) \equiv 1$ ($\bar{Q} = p^{-1}$), the solution, $\Delta(r,t)$, has the formal inverse

$$\Delta(r,t) = \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} \frac{[I_0(qr)K_1(qa) + K_0(qr)I_1(qa)]e^{pt}dp}{kpq[I_1(qb)K_1(qa) - I_1(qa)K_1(qb)]} \quad (9)$$

where the path of integration lies to the right of all singularities of the integrand.

This integral can be evaluated as the sum of its residues at the poles $p = 0$ and $p_n = -\alpha\beta_n^2$, where $\pm\beta_n$ are the roots (real and simple) of

$$J_1(\beta_n b)Y_1(\beta_n a) - J_1(\beta_n a)Y_1(\beta_n b) = 0 \quad (10)$$

for $a > 0$, and

$$J_1(\beta_n b) = 0 \quad (11)$$

for $a = 0$.

The solution for $\Delta(r,t)$ is

$$\Delta(r, t) = \frac{b}{[4k(b^2 - a^2)]}\left[8\alpha t + 2r^2 - b^2 - 3a^2 - 4a^2 \log r + \frac{4a^2}{(b^2 - a^2)}(b^2 \log b - a^2 \log a)\right] + \pi \sum_{n=1}^{\infty} \frac{e^{-\alpha\beta_n^2 t}J_1(\beta_n a)J_1(\beta_n b)[J_0(\beta_n r)Y_1(\beta_n a) - Y_0(\beta_n r)J_1(\beta_n a)]}{k\beta_n[J_1^2(\beta_n a) - J_1^2(\beta_n b)]} \quad (12)$$

If $Q(t)$ is continuous and of exponential order, viz., $\lim_{t\to\infty} Q(t)e^{-pt} = 0$, and $Q'(t)$ is sectionally continuous, then[3]

$$\bar{T}(r,p) = \{\mathcal{L}[Q'(t)] + Q(0)\}\bar{\Delta}(r,p) \quad (13)$$

where $\bar{\Delta}(r,p)$ solves the problem with $Q(t) \equiv 1$.

[Note that Chen[1] omits the derivative sign in his Eq. (19), and printing errors also occur in his Eqs. (13) and (20).]

Use of Eq. (13) and the convolution integral gives the solution

$$T(r,t) = Q(0)\Delta(r,t) + \int_0^t Q'(t - \tau)\Delta(r,\tau)d\tau \quad (14)$$

$$= Q(t)\Delta(r,0) + \int_0^t Q(t - \tau)\frac{\partial\Delta(r,\tau)}{\partial\tau}d\tau \quad (15)$$

The complete solution is then

$$T(r,t) = Q(t)\left\{\frac{b}{4k(b^2 - a^2)}\left[2r^2 - b^2 - 3a^2 - 4a^2 \log r + \frac{4a^2}{b^2 - a^2}(b^2 \log b - a^2 \log a)\right] + \pi \sum_{n=1}^{\infty} \frac{J_1(\beta_n a)J_1(\beta_n b)[J_0(\beta_n r)Y_1(\beta_n a) - Y_0(\beta_n r)J_1(\beta_n a)]}{k\beta_n[J_1^2(\beta_n a) - J_1^2(\beta_n b)]}\right\} + \int_0^t d\tau Q(t - \tau)\left[\frac{2\alpha b}{k(b^2 - a^2)} - \pi \sum_{n=1}^{\infty} \frac{\alpha\beta_n e^{-\alpha\beta_n^2 t}J_1(\beta_n a)J_1(\beta_n b)[J_0(\beta_n r)Y_1(\beta_n a) - Y_0(\beta_n r)J_1(\beta_n a)]}{k[J_1^2(\beta_n a) - J_1^2(\beta_n b)]}\right] \quad (16)$$

Although the case $a = 0$ can be derived as the limit of Eq. (16), it probably is easier to rederive the solution from the beginning; it is

$$T(r,t) = \frac{Q(t)b}{k}\left[\frac{r^2}{2b^2} - \frac{1}{4} - 2\sum_{n=1}^{\infty} \frac{J_0(\beta_n r)}{\beta_n^2 b^2 J_0(\beta_n b)}\right] + \int_0^t Q(t - \tau)\left[\frac{2\alpha}{bk} + 2\sum_{n=1}^{\infty} \frac{\alpha}{bk} e^{-\alpha\beta_n^2\tau}\frac{J_0(\beta_n r)}{J_0(\beta_n b)}\right]d\tau \quad (17)$$

where

$$J_1(\beta_n b) = 0 \quad (18)$$

These equations are valid for all values of $t(\geqslant 0)$ but converge rapidly for large values of t only. Often a rapidly convergent series (or asymptotic series) is required for small values of time and possibly for $r \approx b$; in this case, use of the foregoing formula would be tedious. A more convenient expansion can be obtained by using the asymptotic expressions for the Bessel functions in Eq. (8) for large values of p (and q) corresponding to small values of t. The resulting transforms can be evaluated using the result[2]

$$\mathcal{L}^{-1}\left(\frac{e^{-qx}}{p^{n/2+1}}\right) = (4t)^{n/2}\, i^n \operatorname{erfc}\left[\frac{x}{2(kt)^{1/2}}\right] \quad (19)$$

where

$$i^n \operatorname{erfc} x = \int_x^\infty i^{n-1} \operatorname{erfc} x dx \quad (20)$$

and

$$i^0 \operatorname{erfc} x = \operatorname{erfc} x \quad (21)$$

The solution for $Q(t) \equiv 1$ for small values of t, provided that r/b is not very small, is

$$\Delta_1(r,t) \backsim \frac{1}{k}\left(\frac{b}{r}\right)^{1/2}\left\{2(\alpha t)^{1/2}\, i \operatorname{erfc}\left[\frac{b - r}{2(\alpha t)^{1/2}}\right] + \frac{\alpha t(b + 3r)}{2br}\, i^2 \operatorname{erfc}\left[\frac{b - r}{2(\alpha t)^{1/2}}\right] + \frac{(9b^2 + 33r^2 + 6br)}{128b^2r^2} \times (4\alpha t)^{3/2}\, i^3 \operatorname{erfc}\left[\frac{b - r}{2(\alpha t)^{1/2}}\right]\ldots\right\} + \frac{1}{k}\left\{2\left(\frac{\alpha t b}{r}\right)^{1/2} \times \left[i \operatorname{erfc}\left(\frac{b + r - 2a}{2(\alpha t)^{1/2}}\right)\ldots + i \operatorname{erfc}\left(\frac{3b - r - 2a}{2(\alpha t)^{1/2}}\right) + \ldots + i \operatorname{erfc}\left(\frac{3b + r - 4a}{2(\alpha t)^{1/2}}\right) + \ldots\right]\right\} \quad (22)$$

The corresponding value of T, (r,t) for arbitrary heat flux $Q(t)$, valid for small values of (tb/r), is obtained from Eq. (15), viz.,

$$T_1(r,t) \backsim \frac{1}{k}\int_0^t d\tau\, Q(t - \tau)\left(\frac{b}{r}\right)^{1/2}\left\{\frac{b - r}{2\tau} \times \operatorname{erfc}\left[\frac{b - r}{2(\alpha\tau)^{1/2}}\right] + \left(\frac{\alpha}{\tau}\right)^{1/2}\left(\frac{b^2 + 10br - 3r^2}{8br}\right) \times i \operatorname{erfc}\left[\frac{b - r}{2(\alpha\tau)^{1/2}}\right] + \alpha\left(\frac{9b^3 + 29b^2r + 123br^2 - 33r^3}{64b^2r^2}\right) \times i^2 \operatorname{erfc}\left[\frac{b - r}{2(\alpha\tau)^{1/2}}\right]\ldots + \left(\frac{b + r - 2a}{2\tau}\right) \times \operatorname{erfc}\left[\frac{b + r - 2a}{2(\alpha\tau)^{1/2}}\right] + \frac{3b - r - 2a}{2\tau}\operatorname{erfc}\left[\frac{3b - r - 2a}{2(\alpha\tau)^{1/2}}\right]\right\} \quad (23)$$

This form for the temperature is particularly useful for evaluating the maximum temperature gradients and thermal stresses that often occur for small values of tb/r for impulse-type heat fluxes.

The corresponding formula for $a = 0$ has the basic solution[2] $\Delta_2(r,t)$ with

$$\Delta_2(r,t) \backsim \frac{1}{k}\left[2\left(\frac{\alpha bt}{r}\right)^{1/2} i \text{ erfc}\left(\frac{b-r}{2(\alpha t)^{1/2}}\right) + \frac{\alpha t(b+3r)}{2(br^3)^{1/2}} i^2 \text{ erfc}\left(\frac{b-r}{2(\alpha t)^{1/2}}\right)\ldots\right] \quad (24)$$

and $T_2(r,t) = T_1(r,t)$ to this order of accuracy.

It will be observed, comparing Eq. (22) and Eq. (24), that the first terms of this expansion are independent of the inner radius of the cylinder. Physically, this shows that, for the initial temperature changes only, the reflection of the "temperature wave" at the inner boundary may be neglected, and the temperature may be obtained as if the cylinder were solid.

As time goes on, however, this approximation will get worse, but for impulse-type heat fluxes it should be sufficiently accurate to predict maximum heating rates. The first form for T [Eq. (14)] is useful when $Q'(t - \tau)$ is an impulse-type function, so that $Q(t - \tau)$ is a step function.

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An Alternate Interpretation of Newton's Second Law

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Introduction

THE derivation of the momentum equation for variable mass systems has been reopened recently for discussion. In these various discussions[1] two assumptions usually are made: 1) the classical momentum equations do not apply to systems of variable mass; and 2) the derivation of the rocket equation requires two separate control volumes containing a total constant mass.

In this paper it will be shown that, although assumption 2 is a result of assumption 1, assumption 2 denies assumption 1. It will be shown that the classical momentum equation for a particle is given in an incomplete form and that Newton's equation (in the classical expression) does lead to the rocket equation.

Assumption 1 thus will be shown to be unnecessary, and assumption 2 will be replaced by a calculus operation. A general definition of momentum, valid for Lebesgue measurable mass,[2] will be developed. The new definition of momentum will be shown to be valid for point masses, summations of mass points, piecewise continuous masses, continuous masses, and for both time variable and time invariant masses.

Standard Derivation of "Rocket" Equation

It is customary to define the law of linear momentum in terms of the mass m_i, velocity $\mathbf{v}_i$, and the net force $\mathbf{F}_i$, acting on the ith particle of a system of particles:[3]

$$\mathbf{F}_i = m_i(d\mathbf{v}_i/dt) = (d/dt)(m_i\mathbf{v}_i) \quad (1)$$

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For a collection of n particles, Newton's second law becomes

$$\Sigma_i \mathbf{F}_i = \Sigma_i(d/dt)(m_i \mathbf{v}_i)$$

If the number of particles is time constant, the summation and the differentiation can be interchanged

$$\Sigma_i \mathbf{F}_i = (d/dt)\Sigma_i m_i\mathbf{v}_i = d\mathbf{G}/dt \quad (2)$$

in which $\mathbf{G}$ is the total linear momentum.

Obviously, the derivation of Eq. (2) requires $\mathbf{G}$ to be the momentum of a system of constant mass. It is possible, however, to use Eq. (2) on variable mass system by the following artifice.

Let there be two volumes Y and Y_E having a common boundary. Let Y be the control volume for which the momentum $\mathbf{G}$ is to be found, and let the momentum of the particles in Y_E be $\mathbf{G}_E$. If it is assumed further that the total mass in the two volumes is a constant, then Eq. (2) applies:

$$\Sigma_i \mathbf{F}_i = (d/dt)(\mathbf{G} + \mathbf{G}_E) = (d\mathbf{G}/dt) + (d\mathbf{G}_E/dt) \quad (3)$$

The second term on the right of Eq. (3) seems to imply that forces acting on Y_E should have an effect on Y. This physically indefensible result can be removed by letting the volume Y_E approach zero. In the limit, $d\mathbf{G}_E/dt$ simply becomes the rate at which momentum crosses the boundary of Y. Equation (3) is the expression for the momentum theorem of variable mass systems.

Equation (3) can be applied to a single particle that enters Y at time T_1 and leaves at time T_2:

1) For time $t < T_1$, Eq. (3) is identically zero.
2) For time $t > T_2 > T_1$, Eq. (3) is identically zero.
3) At times T_1 and T_2, $d\mathbf{G}/dt = md\mathbf{v}/dt$, and $d\mathbf{G}_E/dt$ represents an impulsive change of momentum on the boundary.

If one defines $u_1(t - T_1)$ as a unit step function open on the left and $u_2(t - T_2)$ a unit step function open on the right, Eq. (3) becomes for a single particle

$$[u_1(t-T_1) - u_2(t-T_2)]\mathbf{F}_i = [u_1(t-T_1) - u_2(t-T_2)]m_i(d\mathbf{v}_i/dt) + \delta(t-T_1)m_i\mathbf{v}_i - \delta(t-T_2)m_i\mathbf{v}_i \quad (4)$$

in which $\delta(t - T)$ is the Dirac Delta or unit impulse function.[4] It is well known that

$$(d/dt)[u(t-T)] = \delta(t-T)$$

Thus Eq. (4) becomes

$$A_i(T_1,T_2,t)\mathbf{F}_i = (d/dt)[A_i(T_1,T_2,t)m_i\mathbf{v}_i] \quad (5)$$

in which A_i, the closed pulse function, is defined as

$$A_i(T_1,T_2,t) = \begin{cases} 0 & t < T_1 \\ 1 & T_2 \geqslant t \geqslant T_1 \\ 0 & t > T_2 \end{cases}$$

Now the function A_i is nonzero only for particles in the control volume Y; thus one can write the equation for a system of particles by summing over all particles in Y and all particles outside of Y. (This is the division into Y and Y_E, except that now Y_E can be any volume sufficiently large to contain all particles that will be in Y_E at any time and any other additional particles. Y_E could, for example, contain all the particles in the universe except for the particles in Y.)

Thus the momentum equation becomes

$$\sum_{i=1}^{\infty} A_i\mathbf{F}_i = \sum_{i=1}^{\infty}\left[\frac{d}{dt}(A_i m_i\mathbf{v}_i)\right]$$

Here the interchange of derivative and summation is valid

$$\sum_{i=1}^{\infty} A_i\,\mathbf{F}_i = \frac{d}{dt}\left(\sum_{i=1}^{\infty} A_i\, m_i\, \mathbf{v}_i\right)$$

but $A_i = 1$ for particles inside Y and zero for particles out-